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BUCKLING OF CYLINDERS OF SANDWICH CONSTRUCTION IN AXIAL COMPRESSION

No. 1830

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BUCKLING OF CYLINDERS OF SANDWICH CONSTRUCTION IN AXIAL COMPRESSION[1]

By

H. W. MARCH, Mathematician
and
EDWARD W. KUENZI, Engineer

Forest Products Laboratory,[2] Forest Service
U. S. Department of Agriculture

Summary

~~This report presents~~ A theoretical analysis for the behavior of long, circular, cylindrical shells of sandwich construction under axial compressive loads. The analysis is designed to evaluate the effects of the relatively low shearing moduli of sandwich cores on buckling stresses. Families of curves are presented for use in designing shells of sandwich construction having isotropic facings and orthotropic or isotropic cores.

The results of the theoretical analysis were compared with those obtained from tests on a series of curved panels. It was found that the theory applied reasonably well to curved plates of sizes sufficient to include at least one ideal buckle. Application of the theory thus is not limited to long, complete cylinders.

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[1] This progress report is one of a series prepared and distributed by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order No. NAer 01237 and 01202, and U. S. Air Force No. USAF 18 (600)-70. Results here reported are preliminary and may be revised as additional data become available. Original report published June 1952.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Introduction

In the design of aircraft and guided missiles, it was found necessary to devise a method of determining the stress at which curved sandwich panels subjected to axial compression become elastically unstable. It is known that, for thin homogeneous materials, a curved form greatly increases the critical load as compared to a flat sheet of the same approximate size. A similar increase may be expected for curved sandwich panels. Although this report applies primarily to sandwich construction for aircraft, the results are general and apply to any structures of the type considered.

This report presents a theoretical analysis of the behavior of long, circular, cylindrical shells of sandwich construction under axial compressive loads and an experimental confirmation of this analysis by tests on curved panels of sufficient size to include at least one ideal buckle. Thus, these panels are assumed to simulate the action of complete cylinders.

The buckling of a homogeneous, isotropic, thin-walled cylinder was treated by von Karman and Tsien (16)[3] and by Tsien (13, 14) in related papers. These authors assumed, in addition to the wave form of the classical theory, inward buckles of diamond shape to represent the characteristic buckles that are actually observed. They used an energy method to determine the critical compressive stress. This method, in which only diamond-shaped buckles are used, was applied by March (7) to cylinders made of plywood, an orthotropic material. Particular attention was paid to the effect of initial irregularities that contribute to the observed scatter of experimentally determined critical stresses of both isotropic and orthotropic cylinders.

In this report, the effect of shear deformation in the core of a sandwich cylinder is taken into account by employing an approximate "tilting" method. This method was used by Williams, Leggett, and Hopkins in their analysis of flat sandwich panels (18) and by Leggett and Hopkins in their analysis of flat sandwich panels and cylinders (4). It amounts essentially to assuming that the transverse components of shear stress are constant across the thickness of the core. The form of buckles assumed by Leggett and Hopkins (4) in the cylinder is different from that assumed in this report.

The core and facings are taken to be orthotropic, with two of their natural axes parallel, respectively, to the axial and circumferential directions of the cylinder. The facings, which may be equal or unequal in thickness, are

---

[3] Underlined numbers in parentheses refer to Literature Cited at end of report.

assumed to be thin, but their flexural rigidities are not neglected, as these may be of importance in certain cases. All stress components in the core are neglected except the transverse shear components. It was pointed out by Reissner (12) that the stress component in the core normal to the facings may be of importance in the analysis of sandwich shells. Preliminary calculations indicate that the effect of this component is small in the problem under consideration.

As was done in work described by Forest Products Laboratory Report No. 1322-A (7), initial irregularities are assumed to be present and to grow under increasing compressive load until buckling occurs. For a discussion of this important matter, the reader is referred to that report and, in particular, to the observations of the growth of artificially produced initial irregularities. Also as described in report No. 1322-A, a large deflection theory is used to take into account the nonlinear support associated with the curvature of the shell, as discussed by von Karman, Dunn, and Tsien (17). The derivations of the differential equation for a stress function and of the expression for the energy of deformation are extensions of the analysis used by von Karman and Tsien (16) for the homogeneous, isotropic cylinder to the sandwich cylinder composed of orthotropic materials. Suitable modification is made for the effect of shear deformation in the core of the sandwich.

Theoretical Analysis

Choice of Axes Notation

The choice of axes is shown in figure 1, the coordinate y being measured along the circumference. The notations for stress and strain are those of Love's treatise (5). The components of displacement in the axial, circumferential, and radial directions, respectively, are u, v, and w, the latter being positive inward. Since initial irregularities of the cylindrical surface are assumed, the symbol w_o is used to denote the initial distance, measured radially, of a point of the middle surface from a true cylindrical surface of radius r, and the symbol w to denote the corresponding distance at any stage of the deformation. The thickness of the core is denoted by c and that of each facing by f_1 and f_2, respectively.

Extensional Strains and Stresses

Expressions can now be written for the extensional strains uniform across the thickness of the cylindrical shell and for the corresponding mean membrane stresses. On these will be superposed a system of flexural strains, and the energy of deformation associated with each system of strains will be found.

The extensional strains are expressed by the equations:

$$
\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2 \\
e_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2 - \frac{w}{r} + \frac{w_o}{r} \qquad (1) \\
e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w_o}{\partial x}\frac{\partial w_o}{\partial y}
\end{aligned}
$$

In each facing, the corresponding stress components are:

$$
\begin{aligned}
X_x &= \frac{E_x}{\lambda}\left(e_{xx} + \sigma_{yx} e_{yy}\right) \\
Y_y &= \frac{E_y}{\lambda}\left(e_{yy} + \sigma_{xy} e_{xx}\right) \qquad (2) \\
X_y &= \mu_{xy} e_{xy}
\end{aligned}
$$

where $\underline{E_x}$ and $\underline{E_y}$ are Young's moduli, μ_{xy} is the modulus of rigidity for shearing strains referred to the $\underline{x}$ and $\underline{y}$ directions, $\underline{\sigma_{xy}}$ and $\underline{\sigma_{yx}}$ are Poisson's ratios, and $\lambda = 1 - \sigma_{xy}\sigma_{yx}$. All of these quantities are elastic properties of the facings. Because the stress components $\underline{X_x}$, $\underline{Y_y}$, and $\underline{X_y}$ are neglected in the core, the mean membrane stress components for the cylinder are:

$$
\begin{aligned}
\overline{X_x} &= \frac{E_a}{\lambda}\left(e_{xx} + \sigma_{yx} e_{yy}\right) \\
\overline{Y_y} &= \frac{E_b}{\lambda}\left(e_{yy} + \sigma_{xy} e_{xx}\right) \qquad (3) \\
\overline{X_y} &= \mu_m e_{xy}
\end{aligned}
$$

where:

$$
E_a = \frac{E_x (f_1 + f_2)}{h}, \quad E_b = \frac{E_y (f_1 + f_2)}{h}, \quad \mu_m = \frac{\mu_{xy} (f_1 + f_2)}{h} \qquad (4)
$$

and:

$$h = c + f_1 + f_2 \tag{5}$$

From the relation:

$$E_x \sigma_{yx} = E_y \sigma_{xy} \tag{6}$$

that holds for orthotropic materials and equations (4), it follows that:

$$E_a \sigma_{yx} = E_b \sigma_{xy} \tag{7}$$

By using this relation, it is found from equation (3) that:

$$\begin{aligned} e_{xx} &= \frac{1}{E_a}\overline{X_x} - \frac{\sigma_{yx}}{E_b}\overline{Y_y} \\ e_{yy} &= \frac{1}{E_b}\overline{Y_y} - \frac{\sigma_{xy}}{E_a}\overline{X_x} = \frac{1}{E_b}\overline{Y_y} - \frac{\sigma_{xy}}{E_b}\overline{X_x} \\ e_{xy} &= \frac{1}{\mu_m}\overline{X_y} \end{aligned} \tag{8}$$

The mean membrane stress components satisfy the equations of equilibrium:

$$\begin{aligned} \frac{\partial \overline{X}_x}{\partial x} + \frac{\partial \overline{X}_y}{\partial y} &= 0 \\ \frac{\partial \overline{X}_y}{\partial x} + \frac{\partial \overline{Y}_y}{\partial y} &= 0 \end{aligned} \tag{9}$$

They can consequently be expressed in terms of a stress function as follows:

$$\overline{X}_x = \frac{\partial^2 F}{\partial y^2}, \quad \overline{Y}_y = \frac{\partial^2 F}{\partial x^2}, \quad \overline{X}_y = -\frac{\partial^2 F}{\partial x \partial y} \tag{10}$$

It is found from equations (1) by eliminating <u>u</u> and <u>v</u> that:

$$\begin{aligned} \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - \frac{\partial^2 e_{xy}}{\partial x \partial y} &= \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \\ &- \left(\frac{\partial^2 w_o}{\partial x \partial y}\right)^2 + \frac{\partial^2 w_o}{\partial x^2}\frac{\partial^2 w_o}{\partial y^2} - \frac{1}{r}\frac{\partial^2 w}{\partial x^2} + \frac{1}{r}\frac{\partial^2 w_o}{\partial x^2} \end{aligned} \tag{11}$$

By introducing (10) in (8) and substituting the results in (11), the following differential equation for $\underline{F}$ is obtained:

$$A\frac{\partial^4 F}{\partial x^4} + B\frac{\partial^4 F}{\partial y^4} + C\frac{\partial^4 F}{\partial x^2 \partial y^2} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w_o}{\partial x \partial y}\right)^2$$

$$+ \frac{\partial^2 w_o}{\partial x^2}\frac{\partial^2 w_o}{\partial y^2} - \frac{1}{r}\frac{\partial^2 w}{\partial x^2} + \frac{1}{r}\frac{\partial^2 w_o}{\partial x^2} \qquad (12)$$

where:

$$A = \frac{1}{E_b},\ B = \frac{1}{E_a},\ C = \frac{1}{\mu_m} - \frac{2\sigma_{xy}}{E_a} \qquad (13)$$

It is readily established that the following expression represents the energy of extensional deformation of a rectangular portion of the shell with edges of length $\underline{a}$ and $\underline{b}$:

$$W_1 = \frac{h}{2}\int_0^a\int_0^b \left[B\overline{X}_x^{\,2} + A\overline{Y}_y^{\,2} - \frac{2\sigma_{xy}}{E_a}\overline{X}_x\,\overline{Y}_y + \frac{1}{\mu_m}\overline{X}_y^{\,2}\right] dy\,dx \qquad (14)$$

Form of Buckles and Initial Irregularities

The stress components $\overline{X}_x$, $\overline{Y}_y$, and $\overline{X}_y$ in (14) are derived from a stress function $\underline{F}$, satisfying the differential equation (12), which involves derivatives of $\underline{w_o}$ and $\underline{w}$ representing the initial and deformed middle surface of the shell. For $\underline{w}$, the inward radial deflection, the following form will be chosen:

$$\frac{w}{r} = g + \delta \cos^2(\beta y - \alpha x)\cos^2(\beta y + \alpha x) \qquad (15)$$

where

$$\beta = \frac{\pi}{b},\quad \alpha = \frac{\pi}{a} \qquad (16)$$

The nodal lines of the trigonometric portion of equation (15) are shown in figure 2. The displacement $\underline{w}$ is positive inward. In equation (16), $\underline{a}$ and $\underline{b}$ represent the length and width, respectively, of a diamond. The initial irregularities will be assumed to have the form (15). This is done for the purpose of simplifying the calculations. Then $\underline{w_o}$ is chosen in the following form:

$$\frac{w_o}{r} = g_o + \delta_o \cos^2(\beta y - \alpha x)\cos^2(\beta y + \alpha x) \tag{17}$$

An initial flat spot on the surface of the cylinder could be described roughly by equation (17). An initial irregularity was introduced here, as it was in report No. 1322-A (7), to obtain a qualitative description of its influence in causing an isolated buckle to develop in its vicinity. If the initial depth of an irregularity of the form (17) is very small, the dimensions of the area that it occupies are not very important. For this reason in order to simplify the calculations, the dimensions a and b in equation (17) are taken to be the same as those in equation (15). From the qualitative description that is obtained of the development of an isolated buckle, conclusions were drawn in report No. 1322-A that led to the derivation of the final formulas from the analysis for the case $w_o = 0$.

The details of substituting (15) and (17) in (12), of obtaining the stress function F and the stress components $\overline{X}_x$, $\overline{Y}_y$, and $\overline{X}_y$, of substituting these stress components in equation (14), and of related operations are identical with the corresponding operations performed in report No. 1322-A (7). Reference is therefore made to equations (21) and (31) of that report. The following differences in notation should be noted:

Notation of Report No. 1322-A	Notation of Present Report
H	$E_a\ E_b$
$\frac{E_L\ \sigma_{TL}}{H}$	$\frac{\sigma_{yx}}{E_b} = \frac{\sigma_{xy}}{E_a}$
$f,\ f_o$	$\delta,\ \delta_o$
$X',\ Y',\ X_y'$	$\overline{X},\ \overline{Y},\ \overline{X}_y$

From equation (14), the energy of extensional deformation W_1 is then found to be:

$$W_1 = \frac{hab}{8}\left\{4r^4\alpha^4\beta^4(\delta^2 - \delta_o^2)^2\left[\frac{1}{128B\beta^4} + \frac{\left(1 - \frac{1}{r^2\beta^2(\delta+\delta_o)}\right)^2}{128A\alpha^4}\right.\right.$$

$$+ \frac{1}{16(A\alpha^4 + 81B\beta^4 + 9C\alpha^2\beta^2)} + \frac{1}{16(81A\alpha^4 + B\beta^4 + 9C\alpha^2\beta^2)}$$

$$\left.\left. + \frac{4\left(2 - \frac{1}{r^2\beta^2(\delta+\delta_o)}\right)^2 + 1}{64(A\alpha^4 + B\beta^4 + C\alpha^2\beta^2)}\right] + 4Bp^2 + 4Ac_1^2 + \frac{8\sigma_{xy}}{E_a}c_1 p\right\} \tag{18}$$

This equation is equation (31) of report No. 1322-A (7) with the proper values inserted for the former abbreviations M and S. The quantities p and c_1 represent the mean compressive stress and the mean circumferential stress, respectively.

If n is the number of buckles in a circumference, the width b of an individual buckle and n are related by the equation:

$$b = \frac{2\pi r}{n} \tag{19}$$

Then:

$$\beta = \frac{\pi}{b} = \frac{n}{2r} \tag{20}$$

It will be convenient to denote the ratio $\frac{b}{a}$ of the dimensions of a buckle by:

$$z = \frac{b}{a} = \frac{\alpha}{\beta} \tag{21}$$

In the expression obtained from (18) by using equations (20) and (21), let:

$$\eta = n^2\frac{h}{r},\ \xi = \delta\frac{r}{h},\ \xi_o = \delta_o\frac{r}{h} \tag{22}$$

$$K_1 = Az^4 + 81B + 9Cz^2 \tag{23}$$

$$K_2 = 81Az^4 + B + 9Cz^2 \tag{24}$$

$$K_3 = Az^4 + B + Cz^2 \tag{25}$$

$$e_1 = \frac{z^4}{4096B} + \frac{1}{4096A} + \frac{z^4}{512K_1} + \frac{z^4}{512K_2} + \frac{17z^4}{2048K_3} \tag{26}$$

$$e_2 = \frac{1}{512A} + \frac{z^4}{32K_3} \tag{27}$$

$$e_3 = \frac{1}{256A} + \frac{z^4}{32K_3} \tag{28}$$

Equation (18) then becomes:

$$W_1 = hab\left\{\left[e_1\eta^2(\xi^2-\xi_o^2)^2 - e_2\eta(\xi^2-\xi_o^2)(\xi-\xi_o) + e_3(\xi-\xi_o)^2\right]\frac{h^2}{r^2} + \frac{Bp^2}{2} + \frac{Ac_1^2}{2} + \frac{\sigma_{xy}}{E_a}c_1p\right\} \tag{29}$$

Flexural Energy of the Shell

To determine flexural energy of the shell, the following simplified expressions for the changes in curvature and unit twist are used:

$$\frac{\partial^2(w-w_o)}{\partial x^2},\ \frac{\partial^2(w-w_o)}{\partial y^2},\ \frac{\partial^2(w-w_o)}{\partial x\partial y} \tag{30}$$

A discussion of the approximations involved will be found in a paper by Donnell (1). These expressions were used by von Karman and Tsien (16) and by March (7). The expressions (30) are exactly those used in calculating the flexural energy of a flat sandwich plate. The approximate flexural energy of such a plate was found by March (9) and by Ericksen and March (2) by using the "tilting" method of Williams, Leggett, and Hopkins (4, 18).

In this method it is assumed that any line in the core that is initially straight and normal to the undeformed plate will remain straight after the deformation, but will deviate in the x and y directions from the normal to the deformed plate by amounts that are expressed by the parameters k and k'. These parameters are determined by an energy method. These "tilting" factors k and k' are introduced as well as two quantities q and q' that determine the positions of the surfaces in which, respectively, the components u and v of the displacement in the core vanish. The letters k' and q' replace h and r, respectively, of report No. 1583-B, because h and r have already been used in the present report. The following derivation of the expression for the flexural energy follows closely that used for the flat sandwich panel in Forest Products Laboratory Report No. 1583-B (2), to which reference is made for further details. For the sake of simplicity in writing, the initial irregularity w_o will be for the present taken equal to zero. It will then be introduced in the final steps by replacing w by $w - w_o$.

The components of displacement in the core (fig. 3) are taken to be:

$$\begin{aligned} u_c &= -k(\zeta - q)\frac{\partial w}{\partial x} \\ v_c &= -k'(\zeta - q')\frac{\partial w}{\partial y} \\ w_c &= w(x, y) \end{aligned} \tag{31}$$

Thus $\zeta = q$ denotes the surface in which the components of displacement in the $\underline{x}$ direction vanish and $\underline{k}$ is the parameter describing the inclination in the $\underline{x}$ direction of the respective plane sections to the normal to the deformed surface. Similarly $\underline{q'}$ and $\underline{k'}$ are related to the displacements in the $\underline{y}$ direction. These four quantities are to be determined in such a way that the flexural energy associated with a prescribed deflection $\underline{w}$ is a minimum.

To arrive at expressions for the components of displacement in the facings, it is noted that the continuity of the displacement at the facing-to-core bonds requires that the components (31), evaluated at $\zeta = 0$ and $\zeta = c$, shall be those at the inner surfaces of the facings f_1 and f_2, respectively. Within each facing, the components of displacement are assumed to be such that a straight line initially normal to the undeformed surface of the plate will be straight and normal to the deformed surface. Accordingly, the components of displacement in the facings $\underline{f_1}$ and $\underline{f_2}$, respectively, are:

$$
\begin{aligned}
u_1 &= (kq - \zeta)\frac{\partial w}{\partial x} \\
v_1 &= (k'q' - \zeta)\frac{\partial w}{\partial y} \\
w_1 &= w\ (x,\ y)
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
u_2 &= -\left[k\ (c - q) + \zeta - c\right]\frac{\partial w}{\partial x} \\
v_2 &= -\left[k'\ (c - q') + \zeta - c\right]\frac{\partial w}{\partial y} \\
w_2 &= w\ (x,\ y)
\end{aligned}
\tag{33}
$$

The components of strain in the core $\underline{c}$ and facings $\underline{f_1}$ and $\underline{f_2}$ will be denoted by the superscripts $\underline{c}$, 1, and 2, respectively.

From (31), the transverse shear strains in the core are:

$$
e^{(c)}_{\zeta x} = (1 - k)\frac{\partial w}{\partial x} \quad e^{(c)}_{y\zeta} = (1 - k')\frac{\partial w}{\partial y} \tag{34}
$$

The effect of the remaining strains in the core is assumed to be negligible.

In finding the strain energy of the facings in the bending of the plate (or shell), it is convenient to consider the components of strain in the facings to result from the superposition of two states of strain. The first of these consists of the membrane strains in the facings associated with flexure, that is the strain in their middle surfaces. From (32) and (33), these strains are found to be:

$$e_{xx}^{(1)} = (kq + \frac{f_1}{2}) \frac{\partial^2 w}{\partial x^2}$$

$$e_{yy}^{(1)} = (k'q' + \frac{f_1}{2}) \frac{\partial^2 w}{\partial y^2}$$

$$e_{xy}^{(1)} = (kq + k'q' + f_1) \frac{\partial^2 w}{\partial x \partial y} \qquad (35)$$

and

$$e_{xx}^{(2)} = - \left[k\ (c - q) + \frac{f_2}{2} \right] \frac{\partial^2 w}{\partial x^2}$$

$$e_{yy}^{(2)} = - \left[k'\ (c - q') + \frac{f_2}{2} \right] \frac{\partial^2 w}{\partial y^2}$$

$$e_{xy}^{(2)} = - \left[k\ (c - q) + k'\ (c - q') + f_2 \right] \frac{\partial^2 w}{\partial x \partial y} \qquad (36)$$

The second state of strain in the facings is that associated with their bending about their own middle surfaces. This state, in either facing, has the components:

$$e'_{xx} = - \zeta' \frac{\partial^2 w}{\partial x^2}, \quad e'_{yy} = - \zeta' \frac{\partial^2 w}{\partial y^2}, \quad e'_{xy} = - 2\zeta' \frac{\partial^2 w}{\partial x \partial y} \qquad (37)$$

where $\underline{\zeta'}$ is measured from the middle surface of the facing under consideration.

The strain energy in the core or facings is given by the expression (6, 8).

$$U = \frac{1}{2\lambda} \iiint \Big[E_x\, e^2{}_{xx} + E_y\, e^2{}_{yy} + 2E_x\, \sigma_{yx}\, e_{xx} e_{yy}$$

$$+ \lambda\mu_{xy}\, e^2{}_{xy} + \lambda\mu_{y\zeta}\, e^2{}_{y\zeta} + \lambda\mu_{\zeta x}\, e^2{}_{\zeta x} \Big] d\zeta\ dy\ dx \qquad (38)$$

where for the material under consideration (core or facing), $\lambda = 1 - \sigma_{xy}\sigma_{yx}$; $\underline{E_x}$ and $\underline{E_y}$ are Young's moduli, $\underline{\mu_{xy}}$, $\underline{\mu_{y\zeta}}$, and $\mu_{\zeta x}$ are moduli of rigidity; and σ_{xy} and σ_{yx} are Poisson's ratios. Primed letters will denote the elastic constants of the core material and unprimed letters will denote those of the facing material. The integration indicated in formula (38) is to be carried out over the area OABC of figure 2 and the thickness of the core or facings.

The energy in the core is obtained by substituting expressions (34) into (38), the remaining strains in the latter formula being neglected as previously stated. After integrating with respect to ζ over the thickness of the core, the expression for the energy, denoted by U_c, is

$$U_c = \frac{c}{2} \int_0^a \int_0^b \left[\mu'_{\zeta x} \; (1 - k)^2 \left(\frac{\partial w}{\partial x}\right)^2 + \mu'_{y\zeta} \; (1 - k')^2 \left(\frac{\partial w}{\partial y}\right)^2 \right] dy \; dx \qquad (39)$$

The strain energy in the facings associated with the membrane strains is the sum of the energies obtained from (35) and (36). With the substitution of these expressions into (38) one obtains, after integration with respect to ζ, the following expression which is denoted by U_M:

$$U_M = \frac{1}{2\lambda} \int_0^a \int_0^b \left[E_x \left\{ f_1 \left(kq + \frac{f_1}{2}\right)^2 + f_2 \left(k\,(c - q) + \frac{f_2}{2}\right)^2 \right\} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right.$$

$$+ E_y \left\{ f_1 \left(k'q' + \frac{f_1}{2}\right)^2 + f_2 \left(k'\,(c - q') + \frac{f_2}{2}\right)^2 \right\} \left(\frac{\partial^2 w}{\partial y^2}\right)^2$$

$$+ 2E_x \sigma_{yx} \left\{ f_1 \left(kq + \frac{f_1}{2}\right)\left(k'q' + \frac{f_1}{2}\right) + f_2 \left(k\,(c - q) + \frac{f_2}{2}\right)\left(k'\,(c - q') + \frac{f_2}{2}\right) \right\} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$$

$$\left. + \lambda \mu_{xy} \left\{ f_1 \,(kq + k'q' + f_1)^2 + f_2 \left(k\,(c - q) + k'\,(c - q') + f_2\right)^2 \right\} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dy \; dx \quad (40)$$

The strain energy in the facings associated with the flexural strain, U_F, is obtained by substituting expressions (37) into (38) and integrating over the volume of each facing. After integrating with respect to ζ,

$$U_F = \left(\frac{f_1^{\,3} + f_2^{\,3}}{24\lambda}\right) \int_0^a \int_0^b \left[E_x \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + E_y \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2E_x \sigma_{yx} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right.$$

$$\left. + \lambda \mu_{xy} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dy \; dx \qquad (41)$$

Now in all of the expressions (39), (40), and (41) replace $\underline{w}$ by $\underline{w - w_o}$. The flexural energy $\underline{W_2}$ of the region OABC of the shell is the sum of $\underline{U_M}$, $\underline{U_F}$, and $\underline{U_c}$.

For equilibrium, the "tilting" factors $\underline{k}$ and $\underline{k'}$ and the ordinates $\underline{q}$ and $\underline{q'}$ of the neutral surfaces are to be chosen so that the total energy is a minimum. But these factors appear only in the flexural energy $\underline{W_2}$. Hence, they must be chosen to satisfy the conditions

$$\frac{\partial W_2}{\partial(kq)} = 0, \quad \frac{\partial W_2}{\partial(k'q')} = 0, \quad \frac{\partial W_2}{\partial k} = 0, \quad \frac{\partial W_2}{\partial k'} = 0$$

By proceeding exactly as in report No. 1583-B (2), the quadratic form (A14) of that report with $\underline{k'}$ and $\underline{q'}$ replacing $\underline{h}$ and $\underline{r}$, respectively, is obtained for $\underline{W_2}$. The coefficients $\underline{B_i}$ in equation (A14) are defined by equations (A15) in terms of the quantities $\underline{A_i}$, which are defined by:

$$A_1 = \frac{1}{\lambda} \int_0^a \int_0^b \left[E_x \left(\frac{\partial^2 (w - w_o)}{\partial x^2} \right)^2 + \lambda \mu_{xy} \left(\frac{\partial^2 (w - w_o)}{\partial x \partial y} \right)^2 \right] dy\, dx \qquad (42)$$

$$A_2 = \frac{1}{\lambda} \int_0^a \int_0^b \left[E_x \, \sigma_{yx} \frac{\partial^2 (w - w_o)}{\partial x^2} \frac{\partial^2 (w - w_o)}{\partial y^2} + \lambda \mu_{xy} \left(\frac{\partial^2 (w - w_o)}{\partial x \partial y} \right)^2 \right] dy\, dx \qquad (43)$$

$$A_3 = \frac{1}{\lambda} \int_0^a \int_0^b E_y \left(\frac{\partial^2 (w - w_o)}{\partial y^2} \right)^2 + \lambda \mu_{xy} \left(\frac{\partial^2 (w - w_o)}{\partial x \partial y} \right)^2 \, dy\, dx \qquad (44)$$

$$A_4 = \int_0^a \int_0^b \mu'_{\zeta x} \left(\frac{\partial (w - w_o)}{\partial x} \right)^2 dy\, dx \qquad (45)$$

$$A_5 = \int_0^a \int_0^b \mu'_{y\zeta} \left(\frac{\partial (w - w_o)}{\partial y} \right)^2 dy\, dx \qquad (46)$$

On factoring out the common factor, it is found that:

$$2W_2 = r^2 (\delta - \delta_o)^2 ab \Big[B_1' (kq)^2 + 2B_2' (kq)(k'q') + B_3' (k'q')^2$$

$$+ 2B_4' (kq) k + 2B_5' (kq) k' + 2B_5' (k'q') k + 2B_6' (k'q') k'$$

$$+ B_7' k^2 + 2B_8' kk' + B_9' k'^2 + 2B_{10}' (kq) + 2B_{11}' (k'q')$$

$$+ 2B_{12}' k + 2B_{13}' k' + B_{14}' + B_{15}' \Big] \tag{47}$$

where the quantities B_i' are defined in terms of the quantities A_i' by equations (A15) of report No. 1583-B, each A_i' replacing the corresponding A_i in those equations. The quantity in brackets in equation (47) corresponds to $2U'$ in report No. 1583-B. (Note that equation (A22) of that report should read $P = 2U'$).

It is easy to see that the steps of imposing the conditions

$$\frac{\partial W_2}{\partial (kq)} = 0, \quad \frac{\partial W_2}{\partial (k'q')} = 0, \quad \frac{\partial W_2}{\partial k} = 0, \quad \frac{\partial W_2}{\partial k'} = 0$$

and of determining kq, $k'q'$, k, and k' and substituting their values in the expression (47) for $2W_2$ are identical with those taken in report No. 1583-B (2) and that $2W_2$ is equal to the right-hand member of equation (A25) of that report multiplied by $r^2 (\delta - \delta_o)^2$ ab. It is concluded from equations (A26), (A27), and (A28) of report No. 1583-B that:

$$W_2 = 1/2\, r^2 (\delta - \delta_o)^2 ab \left\{ \frac{I \left[A_1' + 2A_2' + A_3' + (A_1'A_3' - A_2'^2)\left(\frac{\phi}{A_4'} + \frac{\phi}{A_5'}\right) \right]}{1 + \frac{A_1'\phi}{A_4'} + \frac{A_3'\phi}{A_5'} + \frac{\phi^2 (A_1 A_3' - A_2'^2)}{A_4' A_5'}} + I_f (A_1' + 2A_2' + A_3') \right\} \tag{48}$$

where I, I_f, and ϕ are defined by:

$$I = \frac{f_1 f_2}{f_1 + f_2} \left(c + \frac{f_1 + f_2}{2} \right)^2 \tag{49}$$

$$I_f = \frac{f_1^3 + f_2^3}{12} \tag{50}$$

$$\phi = \frac{c\, f_1\, f_2}{f_1 + f_2} \tag{51}$$

and $\underline{A_1'}$ by:

$$A_1' = \frac{1}{\lambda} \left(3\, E_x \alpha^4 + \lambda \mu_{xy} \alpha^2 \beta^2\right) \tag{52}$$

$$A_2' = \frac{1}{\lambda} \left(E_x \sigma_{yx} + \lambda \mu_{xy}\right) \alpha^2 \beta^2 \tag{53}$$

$$A_3' = \frac{1}{\lambda} \left(3\, E_y \beta^4 + \lambda \mu_{xy} \alpha^2 \beta^2\right) \tag{54}$$

$$A_4' = \frac{3\alpha^2 \mu'_{\zeta x}}{8}, \qquad A_5' = \frac{3\beta^2 \mu'_{y\zeta}}{8} \tag{55}$$

Note that

$$A_1' + 2A_2' + A_3' = \frac{\beta^4}{\lambda} K_4$$

where

$$K_4 = 3E_x z^4 + 3E_q + 2(E_x \sigma_{yx} + 2\lambda \mu_{xy})\, z^2$$

and introduce the following abbreviations in the expressions for $\underline{A_1'}$, $\underline{A_2'}$, and $\underline{A_3'}$:

$$A_1' = \frac{\beta^4}{\lambda} d_1, \quad A_2' = \frac{\beta^4}{\lambda} d_2, \quad A_3' = \frac{\beta^4}{\lambda} d_3 \tag{56}$$

where:

$$d_1 = 3E_x s^4 + \lambda\mu_{xy} s^2$$

$$d_2 = (E_x \sigma_{yx} + \lambda\mu_{xy}) s^2 \tag{57}$$

$$d_3 = 3E_y + \lambda\mu_{xy} s^2$$

Also,

$$A'_4 = \frac{3\alpha^2 \mu'_{\zeta x}}{8} = \frac{3\beta^2 \mu'_{\zeta x} s^2}{8} \tag{58}$$

Substituting these expressions for A'_i equation (48) becomes

$$W_\zeta = \frac{1}{2\lambda} r^2 (\theta - \theta_0)^2 ab\beta^4 \left\{ \frac{I\left[K_4 + \frac{8\beta^2\phi}{3\lambda}(d_1 d_3 - d_2^2)\left(\frac{1}{\mu'_{\zeta x} s^2} + \frac{1}{\mu'_{y\zeta}}\right)\right]}{1 + \frac{8\beta^2 d_1\phi}{3\lambda\mu'_{\zeta x} s^2} + \frac{8\beta^2 d_3\phi}{3\lambda\mu'_{y\zeta}} + \frac{64\beta^4\phi^2(d_1 d_3 - d_2^2)}{9\lambda^2\mu'_{\zeta x}\mu'_{y\zeta} s^2}} + I_\zeta K_4 \right\} \tag{59}$$

The following transformations are made by using equations (20) and (22):

$$\frac{8\beta^2\phi}{3\lambda\mu'_{\zeta x}} = \frac{2n^2\phi}{3\lambda\mu'_{\zeta x} r^2} = \frac{2\eta\phi}{3\lambda\mu'_{\zeta x} rh} = \frac{\eta}{E_x} S_x$$

where

$$S_x = \frac{2E_x\phi}{3\lambda\mu'_{\zeta x} rh} \tag{60}$$

and

$$\frac{8\beta^2\phi}{3\lambda\mu'_{y\zeta}} = \frac{\eta}{E_x} S_y$$

where

$$S_y = \frac{2E_x\phi}{3\lambda\mu'_{y\zeta} rh} \tag{61}$$

The coefficient of the expression in brackets in equation (59) is also transformed by using equations (20) and (22). The expression for W_2 becomes:

$$W_2 = \frac{(\xi - \xi_o)^2 \, ab \, \eta^2 h^3}{32\lambda r^2} \left\{ \frac{(I/h^3)\left[K_4 + (d_1 d_3 - d_2^2)\frac{\eta}{E_x}\left(\frac{S_x}{z^2} + S_y\right)\right]}{1 + \frac{\eta d_1 S_x}{E_x z^2} + \frac{\eta d_3 S_y}{E_x} + \frac{\eta^2 (d_1 d_3 - d_2^2) S_x S_y}{E_x^2 z^2}} + (I_f/h^3) K_4 \right\} \tag{62}$$

Or

$$W_2 = e_4 \, ab \, \eta^2 \, (\xi - \xi_o)^2 \, \frac{h^3}{r^2} \tag{63}$$

where

$$e_4 = \frac{1}{32\lambda} \left\{ \frac{(I/h^3)\left[K_4 + (d_1 d_3 - d_2^2)\frac{\eta}{E_x}\left(\frac{S_x}{z^2} + S_y\right)\right]}{1 + \frac{\eta d_1 S_x}{E_x z^2} + \frac{\eta d_3 S_y}{E_x} + \frac{\eta^2 (d_1 d_3 - d_2^2) S_x S_y}{E_x^2 z^2}} + (I_f/h^3) K_4 \right\} \tag{64}$$

Virtual Work of the Compressive Load

Exactly as in equation (35) of report No. 1322A (7), the virtual work, W_3, of the compressive load, calculated for the region OABC, figure 2, is found in the notation of the present report to be:

$$W_3 = abh \left[Bp^2 + \frac{\sigma_{xy}}{E_a} pc_1 + \frac{3}{16} r^2 \alpha^2 (\delta^2 - \delta_o^2) p \right]$$

$$= abh \left[Bp^2 + \frac{\sigma_{xy}}{E_a} pc_1 + e_5 \, \eta \, (\xi^2 - \xi_o^2) \, p \, \frac{h}{r} \right] \tag{65}$$

where

$$e_5 = \frac{3}{64} z^2 \tag{66}$$

It will be convenient to consider the mean energy per unit volume of the cylindrical shell. Hence:

$$W = (W_1 + W_2 - W_3)/abh \tag{67}$$

In accordance with equations (29), (63), and (65):

$$W = \left[e_1 \eta^2 (\xi^2 - \xi_o^2)^2 - e_2 \eta (\xi^2 - \xi_o^2)(\xi - \xi_o) + e_3 (\xi - \xi_o)^2 \right.$$

$$\left. + e_4 \eta^2 (\xi - \xi_o)^2 \right] \frac{h^2}{r^2} - e_5 \eta (\xi^2 - \xi_o^2) p \frac{h}{r} - \frac{Bp^2}{2} + \frac{Ac_1^2}{2} \qquad (68)$$

The Buckling Stress

For equilibrium, the derivatives of W with respect to the various parameters g, c_1, ξ, η, and z vanish. From the condition

$$\frac{\partial W}{\partial c_1} = 0, \text{ it follows that } c_1 = 0 \qquad (69)$$

Now c_1 denotes the mean circumferential stress. The parameter g appears only in the expression for c_1 as given by equation (30) of report No. 1322-A (7). The fact that c_1 vanishes implies that g, which describes a uniform radial expansion of the cylinder, takes on such a value that the mean circumferential stress vanishes. Further consideration of the parameter g is not necessary.

From the condition $\frac{\partial W}{\partial \xi} = 0$, it follows that:

$$p = \left[2e_1 \eta (\xi + \xi_o) - \frac{e_2 (3\xi + \xi_o)}{2\xi} + \frac{e_3}{\eta\xi} + \frac{e_4 \eta}{\xi}\right] \frac{(\xi - \xi_o)}{e_5} \frac{h}{r} \qquad (70)$$

where p, as previously noted, is the mean compressive stress.

Let:

$$\gamma_1 = \frac{e_1}{E_a}, \quad \gamma_2 = \frac{e_2}{E_a}, \quad \gamma_3 = \frac{e_3}{E_a}, \quad \gamma_4 = \frac{e_4}{E_a} \qquad (71)$$

Then (70) can be written:

$$p = E_a \left[2\gamma_1 \eta (\xi + \xi_o) - \frac{\gamma_2 (3\xi + \xi_o)}{2\xi} + \frac{\gamma_3}{\eta\xi} + \frac{\gamma_4 \eta}{\xi}\right] \frac{(\xi - \xi_o)}{e_5} \frac{h}{r} \qquad (72)$$

The mean compressive strain ϵ is expressed by:

$$-\epsilon = \frac{1}{ab}\int_0^b dy \left[\frac{1}{a}\int_0^a \frac{\partial u}{\partial x}\, dx\right]$$

As in report No. 1322-A (7), it is found that:

$$\epsilon = Bp + \frac{3\eta}{64} z^2 (\xi^2 - \xi_o^2)\frac{h}{r} \tag{73}$$

The further equilibrium conditions

$$\frac{\partial W}{\partial \eta} = 0 \text{ and } \frac{\partial W}{\partial z} = 0$$

are to be satisfied. The first of these is associated with the number of buckles in a circumference or with the width of an individual buckle, and the second with the ratio b/a of the width of a buckle to its length. It is not analytically feasible to use these conditions in connection with equation (68).

The following method of arriving at the critical value of p is based upon an extended discussion in report No. 1322-A (7). Briefly, it was considered that an isolated initial irregularity would increase in size and depth with increasing mean compressive stress p. It was therefore considered that the load-mean compressive strain curve, with p as a function of ϵ, for a given small initial depth of irregularity would be the envelope of the family of curves for p as a function of ϵ, drawn for a series of values of η by combining equations (71) and (72). On taking into consideration the possibility of jumps from one energy level to another, it was concluded that the critical values of p would scatter considerably, as they actually do in test, depending upon the depth of the initial irregularity and the characteristics of the loading process. It was noted that the value of p at the relative minimum point on the envelope of the curves for p as a function of ϵ, drawn for $\xi_o = 0$, was intermediate among the possible critical values of p. This minimum was accordingly chosen as the "theoretical" critical stress, because it could be conveniently determined by finding a relative minimum of p as a function of ξ and η. It is necessary to employ numerical methods to determine the relative minimum value of p.

In report No. 1322-A (7), the aspect ratio z of the buckles was assumed on the basis of experimental observations before the minimization of p was undertaken. Here, because of the influence of shear deformation in the core, a suitable value to assign to z can not be estimated.

Equation (71), with $\xi_o = 0$, can be written in the form:

$$p = KE_a \frac{h}{r} \tag{74}$$

where

$$K = \frac{32}{3z^2}\left[4\gamma_1\, \eta\xi^2 - 3\gamma_2\xi + \frac{2\gamma_3}{\eta} + 2\gamma_4\, \eta\right] \tag{75}$$

The mean compressive stress, p_f, in the facings is related to the mean compressive stress in the shell by the equation

$$p = \frac{p_f\,(f_1 + f_2)}{h} \tag{76}$$

On recalling the definition of $\underline{E_a}$, it is seen that equation (74) can be written

$$p_f = KE_x\,\frac{h}{r} \tag{77}$$

For a relative minimum of $\underline{p_f}$, the condition

$\frac{\partial K}{\partial \xi} = 0$ must be satisfied.

From this condition, it follows that:

$$\xi = \frac{3\gamma_2}{8\gamma_1\eta} \tag{78}$$

The substitution of this value of $\underline{\xi}$ in (75) yields

$$K = \frac{64}{3z^2}\left(\frac{\gamma_3}{\eta} - \frac{9\gamma_2^{\,2}}{32\gamma_1\eta} + \gamma_4\,\eta\right) \tag{79}$$

In equation (79), $\underline{K}$ is a function of $\underline{n}$ and $\underline{z}$, which occur in the definitions of $\underline{\eta}$, $\underline{\gamma_1}$, $\underline{\gamma_2}$, $\underline{\gamma_3}$, and $\underline{\gamma_4}$. By using the definitions of the quantities $\underline{E_a}$, $\underline{E_b}$, and $\underline{\mu_m}$ that appear through the symbols $\underline{A}$, $\underline{B}$, and $\underline{C}$ in the equations (26), (27), and (28), the following expressions are obtained for $\underline{\gamma_1}$, $\underline{\gamma_2}$, and $\underline{\gamma_3}$ (see equations (71), (26), (27), and (28):

$$\gamma_1 = \frac{z^4}{4096} + \frac{E_y}{4096\,E_x} + \frac{z^4}{512\left(z^4\frac{E_x}{E_y} + 81 + \frac{9z^2E_x}{\mu_{xy}} - 18\sigma_{xy}z^2\right)}$$
$$+ \frac{z^4}{512\left(81z^4\frac{E_x}{E_y} + 1 + 9z^2\frac{E_x}{E_y} - 18\sigma_{xy}z^2\right)} + \frac{17z^4}{2048\left(z^4\frac{E_x}{E_y} + 1 + \frac{z^2E_x}{\mu_{xy}} - 2\sigma_{xy}z^2\right)} \tag{80}$$

$$\gamma_2 = \frac{E_y}{512E_x} + \frac{z^4}{32\left(z^4\frac{E_x}{E_y} + 1 + \frac{z^2E_x}{\mu_{xy}} - 2\sigma_{xy}z^2\right)} \tag{81}$$

$$\gamma_3 = \frac{E_y}{256E_x} + \frac{z^4}{32\left(z^4\frac{E_x}{E_y} + 1 + \frac{z^2E_x}{\mu_{xy}} - 2\sigma_{xy}z^2\right)} \quad . \tag{82}$$

In obtaining the expression for γ_4 from equations (63) and (70), it is convenient to introduce the notation

$$T = 3z^4 + 3\frac{E_y}{E_x} + \frac{2}{E_x}(E_x\sigma_{yx} + 2\lambda\mu_{xy})\,z^2 \tag{83}$$

so that

$$K_4 = E_x\,T \tag{84}$$

and

$$\frac{K_4}{E_a} = \frac{hT}{f_1 + f_2} \tag{85}$$

then

$$\gamma_4 = \frac{T}{32\lambda h^2(f_1 + f_2)}\left[\frac{1 + \frac{1}{K_4}(d_1d_3 - d_2^2)\frac{\eta}{E_x}\left(\frac{S_x}{z^2} + S_y\right)}{1 + \frac{\eta d_1S_x}{E_x z^2} + \frac{\eta d_3S_y}{E_x} + \frac{\eta^2(d_1d_3 - d_2^2)S_xS_y}{E_x^2 z^2}} + I_f\right] \tag{86}$$

Buckling Stress of Sandwich Constructions with Isotropic Facings and Orthotropic or Isotropic Core

For isotropic facings, considerable simplifications can be made. In this case

$$E_x = E_y = (E_x\sigma_{yx} + 2\lambda\mu_{xy}) = E, \quad \mu_{xy} = \mu = E/2\,(1 + \sigma)$$

$$\sigma_{xy} = \sigma_{yx} = \sigma$$

Then

$$\gamma_1 = \frac{1+z^4}{4096} + \frac{z^4}{512\,(z^2+9)^2} + \frac{z^4}{512\,(9z^2+1)^2} + \frac{17\,z^4}{2048\,(1+z^2)^2} \tag{87}$$

$$\gamma_2 = \frac{1}{512} + \frac{z^4}{32\,(1+z^2)^2} \tag{88}$$

$$\gamma_3 = \frac{1}{256} + \frac{z^4}{32\,(1+z^2)^2} \tag{89}$$

$$\gamma_4 = \frac{T}{32\lambda\, h^2\,(f_1+f_2)} \left[\frac{I + \dfrac{I\,(d_1 d_3 - d_2^2)\,\eta}{(3z^4+3+2z^2)\,E^2}\left(\dfrac{S_x}{z^2} + S_y\right)}{1 + \dfrac{\eta d_1 S_x}{E z^2} + \dfrac{\eta d_3 S_y}{E} + \dfrac{\eta^2 S_x S_y\,(d_1 d_3 - d_2^2)}{E^2\, z^2}} + I_f \right] \tag{90}$$

where

$$T = 3z^4 + 3 + 2z^2 \tag{91}$$

$$S_x = \frac{2\,\phi\,E}{3\lambda\,\mu'_{\zeta x} r h} \qquad S_y = \frac{2\,\phi\,E}{3\lambda\,\mu'_{y\zeta} r h} \tag{92}$$

After some manipulation involving substitution of expressions for $\underline{\gamma_1}$, $\underline{\gamma_2}$, $\underline{\gamma_3}$, and $\underline{\gamma_4}$, formula (79) for $\underline{K}$ for sandwich construction with isotropic facings and orthotropic core can be written as:

$$K = \frac{M_1}{\eta} + \frac{2I}{3\lambda\, h^2\,(f_1+f_2)} \left[\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x^2} + \frac{I_f}{I}\; M_2\eta \right] \tag{93}$$

where

$$M_1 = \frac{64}{3z^2}\left(\gamma_3 - \frac{9\gamma_2^2}{32\gamma_1}\right) \tag{94}$$

$$M_2 = \frac{T}{z^2} \tag{95}$$

$$M_3 = \frac{(d_1 d_3 - d_2^2)}{E^2\, z^2}\left(\frac{1}{z^2} + \theta\right) \tag{96}$$

$$M_4 = \left(\frac{d_1}{z^2} + d_3\theta\right)\frac{1}{E} \tag{97}$$

$$M_5 = \frac{(d_1 d_3 - d_2^2)\,\theta}{E^2\, z^2} \tag{98}$$

$$\theta = \frac{S_y}{S_x} \quad \text{or} \quad \theta = \frac{\mu'_{\zeta x}}{\mu'_{y\zeta}} \tag{99}$$

$$d_1 = 3Ez^4 + \frac{1}{2}(1 - \sigma)\,Ez^2 \tag{100}$$

$$d_2 = E\sigma z^2 + \frac{1}{2}(1 - \sigma)\,Ez^2 \tag{101}$$

$$d_3 = 3E + \frac{1}{2}(1 - \sigma)\,Ez^2 \tag{102}$$

If σ is taken to be $\frac{1}{4}$, then:

$$d_1 = 3Ez^2\left(z^2 + \frac{1}{8}\right) \tag{103}$$

$$d_2 = \frac{5}{8}\,Ez^2 \tag{104}$$

$$d_3 = 3E\left(\frac{z^2}{8} + 1\right) \tag{105}$$

If also $\underline{Y_1}$, $\underline{Y_2}$, $\underline{Y_3}$, and $\underline{T}$ are expressed in terms of z and θ (Eq. 87, 88, 89, and 90) then the following expressions can be used in formula 89:

$$M_1 = \frac{1}{12z^2} + \frac{2z^2}{3(1 + z^2)^2} - \frac{3\left[\frac{1}{64z} + \frac{z^3}{4(1 + z^2)^2}\right]^2}{\frac{1 + z^4}{128} + \frac{z^4}{16(z^2 + 9)^2} + \frac{z^4}{16(9z^2 + 1)^2} + \frac{17z^4}{64(1 + z^2)^2}} \tag{106}$$

$$M_2 = 3z^2 + 2 + \frac{3}{z^2} \tag{107}$$

$$M_3 = \frac{1}{8}(9z^4 + 70z^2 + 9)\left(\frac{1}{z^2} + \theta\right) \tag{108}$$

$$M_4 = \frac{3}{8}\left[8z^2 + 1 + (z^2 + 8)\,\theta\right] \tag{109}$$

$$M_5 = \frac{1}{8}\left[9z^4 + 70z^2 + 9\right]\,\theta \tag{110}$$

For constructions for which the shear deformation in the core is negligible, as it is when $\mu'_{\xi x}$ is very large and θ is finite, S_x may be taken equal to zero. Then expression (93) can be minimized with respect to η, resulting in:

$$K_o = 2\sqrt{\frac{2M_1M_2\,(I + I_f)}{3\lambda\,(f_1 + f_2)\,h^2}} \tag{111}$$

Thus, K_o is a function of M_1 and M_2 and the stiffness of the sandwich. It was found by computation that a relative minimum of $M_1M_2 = 0.24$ occurs at $z = 0.95$. The minimum buckling stress is then proportional to:

$$K_o = \frac{4}{5Q_1} \tag{112}$$

where

$$Q_1 = \sqrt{\frac{\lambda\,(f_1 + f_2)\,h^2}{I + I_f}} \tag{113}$$

By letting $N = \frac{K}{K_o}$, the following expression can be written from equation (89) for constructions having any value of S_x:

$$N = \frac{5M_1Q_1}{4\eta} + \frac{5}{6Q_1\,(1 + Q_2)}\left[\frac{M_2\eta + M_3\eta^2S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x^{\,2}} + Q_2M_2\eta\right] \tag{114}$$

where

$$Q_2 = \frac{I_f}{I} \tag{115}$$

It was found in Forest Products Laboratory Report No. 1505 (10) that the values of $I + I_f$ and I_f can be expressed as follows:

$$I + I_f = \frac{1}{12}\left[h^3 - c^3 - \frac{12cd^2}{1 - c/h}\right] \tag{116}$$

$$I_f = \frac{(h - c)^3}{48} + \frac{(h - c)d^2}{4} \tag{117}$$

where

$$d = \frac{f_1 - f_2}{2} \tag{118}$$

After substituting these expressions in the formulas for $\underline{Q_1}$ and $\underline{Q_2}$ and simplifying $\underline{Q_1}$ and $\underline{Q_2}$ become:

$$Q_1 = \sqrt{\frac{12\lambda\,(1 - c/h)^2}{(1 - c^3/h^3)\,(1 - c/h) - \frac{12cd^2}{h^3}}} \tag{119}$$

$$Q_2 = \frac{\frac{1}{4}\,(1 - c/h)^4 + 3\,(1 - c/h)^2\,\frac{d^2}{h^2}}{(1 - c^3/h^3)\,(1 - c/h) - \frac{1}{4}\,(1 - c/h)^4 - 3(1 - c/h)^2\,\frac{d^2}{h^2} - \frac{12cd^2}{h^3}} \tag{120}$$

In equation (114) $\underline{M_1}$, $\underline{M_2}$, $\underline{M_3}$, $\underline{M_4}$, and $\underline{M_5}$ depend upon $\underline{z}$ and $\underline{\theta}$, and $\underline{Q_1}$ and $\underline{Q_2}$, depend upon $\underline{c/h}$ and $\underline{d/h}$. Formula (114) can then be written with appropriate values of $\underline{\theta}$, $\underline{c/h}$, and $\underline{d/h}$ and then a relative minimum value $\underline{N}$ found by choosing a series of values of $\underline{z}$ and $\underline{\eta}$. The facing stress at which buckling will occur is then given by:

$$p_f = \frac{4N}{5Q_1}\,E\,\frac{h}{\gamma} \tag{121}$$

Buckling Stress of Sandwich Constructions with Isotropic Facings of Equal Thickness and Orthotropic or Isotropic Core

Factors in formula (114) can be simplified for sandwich constructions having facings of equal thickness. Then $d = 0$ and after simplification

$$Q_1 = 2\sqrt{\frac{?\lambda}{c^2/h^2 + c/h + 1}} \tag{122}$$

$$Q_2 = \frac{1}{3}\left(\frac{1 - c/h}{1 + c/h}\right)^2 \qquad (123)$$

and finally formula (114) becomes

$$N = \frac{5Q_1}{4}\left\{\frac{M_1}{\eta} + \frac{(1 + c/h)^2}{24\lambda}\left[\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x^{\,2}}\right] + \frac{(1 - c/h)^2}{72\lambda}\, M_2\eta\right\} \qquad (124)$$

The buckling load, which is proportioned to N is obtained by finding the lowest relative minimum of expression (124) with respect to η and z. Expression (124) can be minimized by taking a derivative with respect to η and setting the derivative equal to zero. This leads to a sixth power equation in η. Minimum roots of η with respect to z and for various values of S_x, c/h, and θ were determined by means of a digital computer. Minimum values of N at various values of S_x and for c/h equal to 0.9, 0.8, 0.7, and θ equal to 0.4, 1.0, and 2.5 are given in Table 1 and shown as functions of S_x in figures 4, 5, and 6. Also included in the table and figures are values of N for c/h = 1. These values represent sandwich constructions for which the stiffness of the individual facings are assumed to be zero. Although no actual constructions can be made of this type, the values can be considered as representing the limit for constructions having extremely thin facings. These values of N were obtained as follows. Substitution of c/h = 1 in equation (124) for N leads to

$$N = \frac{5}{2}\sqrt{\lambda}\left[\frac{M_1}{\eta} + \frac{1}{6\lambda}\left(\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x^{\,2}}\right)\right] \qquad (125)$$

which has one relative minimum value for $\eta = \infty$. This minimum value is given by

$$N = \frac{5M_3}{12\sqrt{\lambda}\, S_x\, M_5} \qquad (126)$$

Substituting in this equation the values of M_3 and M_5 given by equations (108) and (110) yields

$$N = \frac{5\left(\frac{1}{z^2} + \theta\right)}{12\sqrt{\lambda}\; S_x\, \theta} \qquad (127)$$

which is minimum for $s = \infty$. This minimum value (for $\sigma = 1/4$; hence $\lambda = 15/16$) is given by

$$N = \frac{5}{3\sqrt{15}\, S_x} = \frac{0.431}{S_x} \tag{128}$$

Substitution of this value of N in equation (118) and using the value of S_x for $f_1 = f_2$ and $c/h = 1$ leads to the following limiting expression for p_f:

$$p_f = \frac{4Eh}{10\sqrt{\lambda}\, r} \cdot \frac{5}{12\sqrt{\lambda}} \cdot \frac{3\lambda\, r\mu'_{\zeta x}}{Ef} = \frac{h}{2f}\, \mu'_{\zeta x} \tag{129}$$

For values of S_x ranging from 0 to about 0.6 it was found that equation (128) did not give lowest minimum values. In this range of S_x the minimum values were obtained from equation (124) by use of a digital computer.

The value of N given by equation (128) and the value of the stress given by equation (129) are independent of the radius of the cylinder and are the usual critical values associated with shear instability of the core (15).

The value of θ of 0.4 and its reciprocal 2.5 were used in the calculations because they apply to honeycomb cores oriented with the weak direction and the strong direction parallel to the length of the cylinder. It has been noted from figures 4 and 6 that in the range of small values of S_x where N is independent of c/h, the orientation of the core makes little difference in the value of N.

Application of Theoretical Results

The compressive facing stress at which buckling of cylinders of orthrotropic sandwich construction occurs is given by equation (77).

$$p_f = KE_x \frac{h}{r}$$

where E_x is modulus of elasticity of facings in axial direction, h is sandwich thickness, r is mean radius of curvature, and K is given by formula (79) as

$$K = \frac{64}{3s^2}\left[\frac{\gamma_3}{\eta} - \frac{9\gamma_2^2}{32\gamma_1\eta} + \gamma_4\eta\right]$$

where values of γ_1, γ_2, and γ_3 are functions of s according to equations (80), (81), and (82) and γ_4 is a function of η and s according to equation (86) and K is taken as the least relative minimum with respect to η and s.

For sandwich constructions having isotropic facings of unequal thickness and orthotropic core K is given by

$$K = \frac{4N}{5Q_1}$$

where N is given by equation (114) as

$$N = \frac{5M_1Q_1}{4\eta} + \frac{s}{6Q_1(1 + Q_2)}\left[\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x{}^2} + Q_2M_2\eta\right]$$

where Q_1 and Q_2 are given by equations (119) and (120) and M_1, M_2, M_3, M_4, and M_5 are functions of s according to equations (106), (107), (108), (109), and (110) and N is taken as the least relative minimum with respect to η and s.

For sandwich constructions having isotropic facings (Poisson's ratio 1/4) of equal thickness and orthotropic core such that $\theta = 0.4$[4] or $\theta = 2.5$[4] or isotropic core ($\theta = 1.0$) equation (114) for N has been solved for $c/h = 1.0$, 0.9, 0.8, and 0.7. Values of N for various S_x values are given in Table 1 and in graphs in figures 4, 5, and 6. Then the critical facing stress is given by

$$p_f = \frac{4N}{5Q_1} E \frac{h}{r}$$

where

$$Q_1 = \frac{3}{2}\sqrt{\frac{5}{c^2/h^2 + c/h + 1}}$$

and N is given in terms of S_x where

[4]These ratios for θ were chosen as representative of honeycomb cores such as were evaluated in Forest Products Laboratory Report No. 1849.

$$S_x = \frac{16cfE}{45rh\mu'_{\zeta x}}$$

and c is core thickness, f is facing thickness, E is modulus of elasticity of facings, h is sandwich thickness, (c + 2f), r is mean radius of curvature, and $\mu'_{\zeta x}$ is modulus of rigidity of core associated with shear strains in the axial-radial plane.

The graphs can be used with little error for determining N for constructions having facings of unequal thickness, provided S_x is calculated using formula (60) and Q_1 is calculated using equation (119).

The analysis may be extended to apply at stresses greater than the proportional limit stress of the facings by use of an appropriate tangent or reduced modulus of elasticity for the facings. This entails a "trial-and-error" solution involving use of the tangent or reduced modulus in the quantity S and elsewhere until the resultant facing stress is compatible with the stress-modulus curve.

Results of the theoretical analysis fall approximately into three zones, depending upon whether there is no shear deformation in the core ($S_x = 0$), some shear deformation in the core (small values of S_x), or considerable shear deformation in the core (large values of S_x).

For no shear deformation, the buckling stress is determined essentially by means of the isotropic or orthotropic theory (depending upon facing properties) with the stiffness determined by considering the spaced facings of the sandwich.

For large shear deformations, the critical stress is associated with instability of the core in shear. This has been observed for sandwich constructions in general (15), and it has been found that the mean critical stress thus determined is the same, regardless of the original assumption of the buckled shape. The smallest value of S_x at which the critical stress is determined by shear instability of the core, however, is greatly affected by the assumed form of the buckled shape. The inclusion of the stiffnesses of the facings I_f gives rise to the family of curves for different values of c/h, as shown in figures 4, 5, and 6, instead of a single curve. If the stiffnesses of the individual facings had been neglected, one curve only, that for c/h = 1, would have resulted. The percentage increase in buckling stress due to the stiffnesses of the individual facings increases as the shear deformation increases. For small shear deformations, the increase is negligible.

From the reasoning involved in the theory leading to equation (77), considerable scatter in the experimental values of the critical stress is to be expected because of the effect of initial irregularities. Similar scatter is exhibited by homogeneous cylindrical shells, for the same reason.

The possibility of failure by wrinkling of the facings at a stress lower than that predicted by equation (77) should be considered.

The analysis in this report involves a number of approximations and assumptions. Such procedures are necessary until a more rigorous treatment of the problem is developed. A completely rigorous treatment of the buckling of a homogeneous cylindrical shell is still lacking, in spite of the noteworthy contributions of von Karman and Tsien.

Tests of Curved Panels

The large size of complete circular, cylindrical shells having realistic facing and core thicknesses and curvatures could not be adapted to the available testing apparatus. Therefore, axial compressive tests were conducted on rectangular panels curved to various radii. The dimensions of these panels were chosen so that their widths and lengths were large enough to include at least one buckle of a size predicted by theory ($b > \frac{2\pi r}{n}$ and $a > \frac{2\pi r}{zn}$) as shown in table 2.

It was then assumed that the curved panel would behave approximately as a complete cylinder. The type of edge support (described later) was such as to produce no clamping.

Test Specimens

The test specimens were essentially of isotropic construction having facings of clad 24ST aluminum alloy on cores of either balsa wood, oriented so that the grain direction was normal to the facings, or of corkboard of three different densities. Corkboard cores were chosen, because their low moduli of rigidity afforded means of exploring shells in which sizeable reductions of buckling stresses, caused by large core shear deformations, could easily be obtained. These corkboard cores had shearing moduli of 1,500, 950, and 320 pounds per square inch, as compared to 15,000 pounds per square inch for the end-grain, balsa-wood core.

Dimensions of the specimens are given in table 2. The panel sizes ranged from approximately 70 inches square to panels 12 inches wide and 30 inches long. Mean radii of curvature ranged from approximately 90 inches to 10 inches. The sandwich constructions had facings of 0.012 inch, 0.020 inch, or 0.032 inch thickness on cores of approximately 1/8 inch, 1/4 inch, or 1/ inch thickness. All constructions tested had facings of equal thickness.

The specimens were manufactured by the bag-molding process. Detailed description of techniques and bonding adhesives used in this process are given in Forest Products Laboratory Report No. 1574 (3). The curvature was attained at the time of molding by using a steel mold curved to the desired radius. A strip of aluminum 1 inch wide and 0.032 inch thick was bonded to the facings at each end of the specimen. This was done to facilitate machining of the specimen ends and also to prevent local end failure during the test. The ends of the specimens were machined square and true in a milling machine.

Testing

The vertical edges of the specimens were held straight by loose-fitting wood guides. These guides were approximately 2 inches by 2 inches in cross section and of lengths 1/4 inch shorter than the test specimen. They were grooved in the lengthwise direction with grooves approximately 1/4 inch deep and wide enough to allow the guides to be slipped onto the edges of the test specimen. No attempt was made to clamp the vertical edges by fitting the guides tightly.

The lower ends of specimens not wider than 30 inches were placed on a heavy flat plate, which was supported by a spherical bearing placed on the lower head of a hydraulic testing machine. The heads of the testing machine were then brought together until the specimen just touched the upper platen with no load indicated. Adjustments were made on the spherical base until no light could be seen between the ends of the specimen and the loading heads. Screw jacks were then placed under the lower loading plate to prevent tilting of the plate while the load was being applied to the specimen. A single thickness of blotting paper was inserted at the ends of the specimen to help prevent local end failures. The load was then applied slowly until failure occurred.

Specimens wider than 30 inches were tested between the heads of a four-screw, mechanically operated, testing machine. No spherical bearing was used. The specimens were cut as true as possible. If light could be seen between the ends of the specimen and the heads of the testing machine, shims of paper or brass were inserted until the gap was closed. These wide specimens were also very long; therefore, small irregularities in the end bearing were absorbed early in the test without causing large variations from uniformity in the stresses in the facings.

Results of Tests

The facing stresses at the failing loads of the curved panels are given in table 2. For later comparison with theoretical values, the parameter N was calculated for each test specimen by using the formula

$$N = \frac{5Q_1 p_f r}{4Eh}$$

where

E = 10,000,000 pounds per square inch (modulus of elasticity of facings)

The visible failures of the specimens were of a type caused by buckling. Large, thin specimens actually showed large buckles, which disappeared after release of load. The appearance of these buckles always caused a sudden drop in the load. Small, thick specimens showed buckling, followed immediately by a crimping appearance at the edges of the buckle. This crimping was undoubtedly due to shear failure of the core caused by high stresses induced in the sandwich by the buckle. Many of the thick specimens exhibited no visible signs of buckling but showed similar crimping. The rapidity of failure occurring immediately upon buckling undoubtedly prevented visual observation of the buckle itself. Similar behavior was observed for cylindrical shells of plywood (11).

Comparison of Theoretical and Experimental Results

The theoretical and experimental values of N are given in table 2. A comparison between them may be obtained by referring to figure 7 which shows the experimental values plotted against the theoretical values. The scatter of points about a line representing equality between experimental and theoretical values shows that theory and experiment agree within approximately ± 30 percent.

In view of the inevitable scatter of experimentally determined buckling stresses that is associated with initial irregularities of shape and variations of material properties, it is concluded that the agreement between results of tests and of theory is satisfactory. The scatter is not as great for shells of sandwich construction as that observed for thin, homogeneous shells (fig. 44, report No. 1322-A (7)) or for plywood shells (fig. 3, Forest Products Laboratory Report No. 1322 (18)). This reduction in scatter may be attributed to a greater total thickness of shell. Thus, irregularities that depart from the true cylindrical surface of the order of the thickness of the shell are less likely to occur in sandwich shells than in thin, homogeneous shells.

Conclusions

The buckling stress of long, thin-walled, circular cylinders of sandwich construction in axial compression can be found with satisfactory accuracy by the formulas and curves of the approximate theoretical analysis of this report.

Curved panels of sizes large enough to include at least one ideal buckle ($b > \frac{2\pi r}{n}$ and $a > \frac{2\pi r}{sn}$) buckle at stresses approximately equal to those of a long, complete cylinder.

Literature Cited

(1) DONNELL, L. H.
1934. Stability of Thin-Walled Tubes in Torsion. National Advisory Committee on Aeronautics Twentieth Annual Report, pp. 95-116.

(2) ERICKSEN, W. S. and MARCH, H. W.
1950. Effects of Shear Deformation in the Core of a Flat, Rectangular Sandwich Panel. Compressive Buckling of Sandwich Panels Having Facings of Unequal Thickness. Forest Products Laboratory Report No. 1583-B.

(3) HEEBINK, B. G., MOHAUPT, A. A., and KUNZWEILLER, J. J.
1947. Fabrication of Lightweight Sandwich Panels of the Aircraft Type. Forest Products Laboratory Report No. 1574.

(4) LEGGETT, D. M. A. and HOPKINS, H. G.
1942. Sandwich Panels and Cylinders Under Compressive End Loads. British Royal Aircraft Establishment Report No. S. M. E. 3203.

(5) LOVE, A. E. H.
1927. Treatise on the Mathematical Theory of Elasticity. Cambridge University Press.

(6) MARCH, H. W.
1942. Flat Plates of Plywood Under Uniform or Concentrated Loads. Forest Products Laboratory Report No. 1312.

(7) ____________
1943. Buckling of Long, Thin Plywood Cylinders in Axial Compression. Forest Products Laboratory Report No. 1322-A (Supplement to Report No. 1322).

(8) ____________
1944. Stress-Strain Relations in Wood and Plywood Considered as Orthotropic Materials. Forest Products Laboratory Report No. 1503.

(9) ____________
1948. Effects of Shear Deformation in the Core of a Flat, Rectangular Sandwich Panel. Forest Products Laboratory Report No. 1583.

(10) MARCH, H. W. and SMITH, C. B.
1949. Flexural Rigidity of a Rectangular Strip of Sandwich Construction. Forest Products Laboratory Report No. 1505.

(11) ____________, NORRIS, C. B., and KUENZI, E. W.
1943. Buckling of Long, Thin Plywood Cylinders in Axial Compression. Forest Products Laboratory Report No. 1322.

(12) REISSNER, ERIC
1949. Small Bending and Stretching of Sandwich-Type Shells. National Advisory Committee on Aeronautics Technical Note No. 1832.

(13) TSIEN, H. S.
1942. Buckling of a Column with Nonlinear Lateral Supports. Jour. Aero. Sc. Vol. 9, p. 119.

(14) ____________
1942. A Theory for the Buckling of Thin Shells. Jour. Aero. Sc. Vol. 9, p. 373.

(15) U. S. Forest Products Laboratory
1951. Sandwich Construction for Aircraft, Part II. ANC-23 Bulletin Part II, pp. 39, 40, 41. Published by Munitions Board.

(16) VON KARMAN, TH. and TSIEN, H. S.
1941. The Buckling of Thin, Cylindrical Shells Under Axial Compression. Jour. Aero. Sc. Vol. 8, p. 303.

(17) ____________, DUNN, L. G., and TSIEN, H. S.
1940. The Influence of Curvature on the Buckling Characteristics of Structures. Jour. Aero. Sc. Vol. 7, p. 276.

(18) WILLIAMS, D., LEGGETT, D. M. A., and HOPKINS, H. G.
1941. Flat Sandwich Panels Under Compressive End Loads. British Royal Aircraft Establishment Report No. A. D. 3174.

Notation

a	length of buckle.
A	$1/E_b$.
A_1, A_2, A_3, A_4, A_5	defined by equations (42) to (46).
$A'_1, A'_2, A'_3, A'_4, A'_5$	defined by equations (52) to (55).
b	width of buckle.
B	$1/E_a$.
c	thickness of the core.
c_1	mean circumferential stress.
C	$\frac{1}{\mu_m} - \frac{2\sigma_{xy}}{E_a}$
d	$\frac{f_1 - f_2}{2}$
d_1, d_2, d_3	defined by equation (57).
e_{xx}, e_{xy}, etc.	components of strain.
e_1, e_2, e_3	defined by equations (26), (27), and (28).
e_4	defined by equation (64).
e_5	defined by equation (66).
E_x, E_y	Young's moduli of the facings
E_a	$\frac{E_x (f_1 + f_2)}{h}$
E_b	$\frac{E_y (f_1 + f_2)}{h}$
f_1, f_2	thicknesses of the facings.
g	quantity proportional to mean radial expansion.

h	$c + f_1 + f_2$.
I	defined by equation (49).
I_f	defined by equation (50).
k, k'	parameters introduced in equations (31).
K	see equations (74), (75), and (79).
K_1, K_2, K_3	defined by equations (23), (24), and (25).
K_4	defined by equation (84).
$\bar{n}$	$2\pi r/b$.
p	mean compressive stress.
p_f	compressive stress in the facings.
q, q'	introduced in equations (31).
r	radius of middle surface of the cylindrical shell.
S_x, S_y	defined by equations (92).
T	defined by equation (83).
u	axial component of displacement.
v	circumferential component of displacement.
U_c	strain energy of the core in the bending of the sandwich shell.
U_F, U_M	strain energy of the facings in the bending of the sandwich shell.
w	radial component of displacement.
W_1	extensional strain energy.
W_2	flexural strain energy.

W_3	virtual work of the compressive load.
W	$(W_1 + W_2 - W_3)/abh$.
X_x, X_y, etc.	components of stress.
s	b/a.
α	π/a.
β	π/b.
γ_1, γ_2, γ_3, γ_4	defined by equations (80), (81), (82), and (86).
δ	a parameter that is proportional to depth of a buckle.
δ_o	initial value of δ.
ϵ	mean compressive strain.
ζ	coordinate shown in fig. 3.
η	$n^2 h/r$.
λ	$(1 - \sigma_{xy}\,\sigma_{yx})$.
μ_{xy}	modulus of rigidity of the facings.
$\mu'_{\zeta x}$, $\mu'_{y\zeta}$	moduli of rigidity of the core.
μ_m	$\dfrac{\mu_{xy}(f_1 + f_2)}{h}$
ξ	$\delta\, r/h$.
ξ_o	$\delta_o\, r/h$.
σ_{xy}, σ_{yx}	Poisson's ratios of the facings.
ϕ	defined by equation (51).
θ	S_y/S_x.

Table 1.--Buckling of sandwich cylinders in axial compression. $p_f = \frac{4\lambda}{5\lambda_c} E \frac{h}{r}$ N values

S_x	θ = 0.4				θ = 1.0				θ = 2.5			
	c/h = 1.0	c/h = 0.9	c/h = 0.8	c/h = 0.7	c/h = 1.0	c/h = 0.9	c/h = 0.8	c/h = 0.7	c/h = 1.0	c/h = 0.9	c/h = 0.8	c/h = 0.7
0	1.000	1.000	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000	1.000
0.1	.982	.983	.981	.980	.972	.972		.972	.951	.949	.949	.946
.2	.960	.962	.960	.958	.943	.941		.935	.902	.900	.896	.889
.4	.922	.921	.917	.912	.886	.880	[illegible]	.870	.808	.801	.794	.785
.6	.720	.766	.826	.870	.720	.765	[illegible]	.801	.720	.712	.699	.691
.8	.540	.600	.670	.752	.540	.599	.680	.708	.540	.605	.614	.602
1.0	.431	.500	.578	.668	.431	.500	.575	.609	.431	.497	.516	.526
1.2	.358	.431	.514	.611	.358	.431	.506	.540	.358	.426	.449	.466
1.4	.308	.385	.468	.563	.308	.385	.451	.486	.308	.372	.398	.417
1.6	.269	.347	.436	.523	.269	.349	.411	.446	.269	.334	.360	.384
2.0	.216	.296	.387	.460	.216	.298	.350	.386	.216	.276	.306	.332
3.0	.143	.230	.317	.356	.143	.224	.266	.304	.143	.199	.233	.262
4.0	.109	.197	.269	.315	.109	.181	.224	.258	.109	.161	.191	.222
6.0	.073	.161	.215	.258	.073	.138	.175	.210	.073	.120	.151	.185
8.0	.053	.138	.186	.227	.053	.115	.151	.187	.053	.097	.129	.165
10.0	.044	.120	.164	.214	.044	.100	.135	.173	.044	.084	.118	.153

Table 2.--Experimental and theoretical data for curved panels of sandwich construction in axial compression. Cores isotropic

Specimen No.	Facing[1] thickness f	Core thickness c	Sandwich thickness h	Panel width	Panel length	Mean radius of curvature r	Experimental: Facing stress at buckling	Experimental: E	Core shear modulus $G_{\zeta x}$	S_x	c/h	Computed values: E	Computed values: z[2]	Computed values: η[2]	Computed values: Buckle width b[3]	Computed values: Buckle length a[3]
	In.	In.	In.	In.	In.	In.	P.s.i.		P.s.i.							
1155-25	0.012	0.256	0.280	12.1	29.3	22.7	13,100	0.268	950	1.11	0.915	0.306	....	....	....	....
1155-21	.020	.310	.350	20.1	29.9	18.5	14,980	.127	950	5.76	.928	.176	50	0.025	....	....
9114-25	.012	.263	.287	12.1	29.9	25.4	6,660	.137	320	5.12	.916	.156	81	.007	....	....
1155-18	.020	.511	.551	12.1	30.0	11.2	16,190	.082	950	6.20	.928	.122	1.9	6.94	5.92	3.12
9114-17	.020	.516	.556	12.1	29.9	27.8	9,330	.100	320	8.66	.928	.097	1.7	7.19	8.52	5.01
9114-21	.020	.515	.555	20.1	29.7	18.8	9,590	.081	320	10.90	.929	.082	1.6	7.34	7.47	4.67
9114-18	.020	.508	.548	12.0	29.9	11.6	11,470	.061	320	17.76	.927	.062	1.4	7.21	5.89	4.21
ABC 113-1	.020	.484	.524	12.1	30.0	11.0	11,200	.059	320	18.14	.924	.061	1.4	6.96	5.72	4.09
ABC 113-2	.020	.500	.540	12.1	30.0	10.9	11,810	.060	320	18.05	.926	.061	1.4	7.11	5.78	4.09
61-1	.012	.133	.157	69.2	70.0	98.9	7,100	1.102	15,000	.0186	.848	.994	.95	.159	60.2	63.4
61-3	.012	.129	.153	69.2	69.9	73.2	8,300	1.046	15,000	.0183	.844	.990	.96	.160	52.5	54.7
1463-10	.020	.262	.302	11.8	29.2	22.5	12,510	.242	1,500	1.02	.868	.343	2.5	3.08	9.29	3.72
1463-14	.020	.259	.299	20.1	29.4	19.3	13,070	.219	1,500	2.12	.866	.307	2.2	3.56	8.25	3.74
1155-10	.020	.254	.294	12.1	30.0	24.4	11,660	.252	950	2.29	.864	.265	2.0	3.55	8.91	4.45
9114-15	.020	.262	.302	20.0	29.9	65.4	4,740	.270	320	2.70	.868	.245	1.9	3.72	14.56	7.66
1155-14	.020	.255	.295	20.1	29.9	19.4	10,320	.176	950	3.14	.864	.224	1.8	3.69	7.79	4.33
1463-11	.020	.262	.302	11.8	29.4	10.6	16,810	.153	1,500	3.30	.868	.200	1.7	3.87	5.72	3.36
1155-11	.020	.252	.292	12.1	29.9	10.5	12,380	.115	950	6.15	.863	.150	1.5	3.81	5.61	3.74
9114-10	.020	.249	.289	12.0	29.7	26.2	6,680	.151	320	7.10	.861	.135	1.4	3.76	8.92	6.37
9114-14	.020	.257	.297	20.1	29.6	18.7	6,770	.110	320	10.30	.865	.110	1.3	3.86	7.47	5.75
9114-11	.020	.257	.297	12.1	30.0	11.5	8,420	.085	320	17.30	.865	.085	1.2	3.85	5.89	4.91
1463-25	.032	.264	.328	11.8	29.4	22.9	12,170	.229	1,500	2.57	.805	.286	1.7	2.57	10.74	6.31
1155-25	.032	.256	.320	12.0	29.9	23.2	12,120	.257	950	4.12	.800	.218	1.5	2.58	10.68	7.12
9114-25	.032	.261	.325	12.1	29.8	27.2	7,480	.168	320	10.43	.804	.130	1.2	2.63	11.49	9.58
1155-6	.020	.145	.185	20.1	29.9	60.5	6,370	.563	950	.77	.784	.594	2.2	1.88	15.26	6.94
1463-5	.020	.134	.174	11.8	29.4	23.6	9,600	.354	1,500	1.34	.770	.453	1.8	2.04	8.92	4.96
1463-7	.020	.135	.175	20.1	29.3	19.0	9,020	.268	1,500	1.32	.772	.370	1.7	2.13	7.85	4.62
1155-5	.020	.137	.177	12.1	29.9	27.7	7,220	.340	950	2.39	.774	.349	1.7	2.17	9.42	5.54
9114-6	.020	.128	.168	20.0	29.8	58.6	3,310	.319	320	2.39	.762	.287	1.5	2.12	13.50	9.00
1155-7	.020	.135	.175	20.1	29.9	18.6	8,340	.245	950	3.11	.772	.270	1.5	2.25	7.54	5.05
1463-4	.020	.135	.175	11.8	29.4	10.5	13,670	.225	1,500	3.43	.772	.252	1.5	2.24	5.71	3.80
1155-4	.020	.135	.175	12.1	30.0	10.8	10,450	.177	950	5.35	.772	.196	1.5	2.25	5.75	4.42
9114-3	.020	.135	.175	12.1	30.0	26.0	5,620	.230	320	6.39	.772	.176	1.3	2.27	8.85	6.81
9114-7	.020	.125	.165	20.1	29.8	16.4	5,820	.179	320	9.15	.759	.154	1.2	2.15	7.47	6.25
9114-4	.020	.145	.185	12.1	29.9	11.5	8,580	.149	320	15.35	.781	.118	1.1	2.55	5.92	5.38

[1] All facings were of equal thickness and of clad 2024-T3 aluminum alloy.

[2] These values of z and η produced minimum theoretical values of E.

[3] $b = 2\pi\sqrt{\frac{rh}{\eta}}$ $a = \frac{b}{z}$

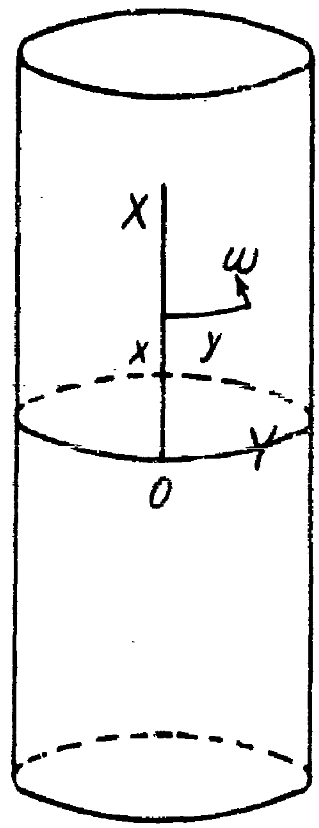

Figure 1. --Choice of coordinates on the surface of a cylinder.

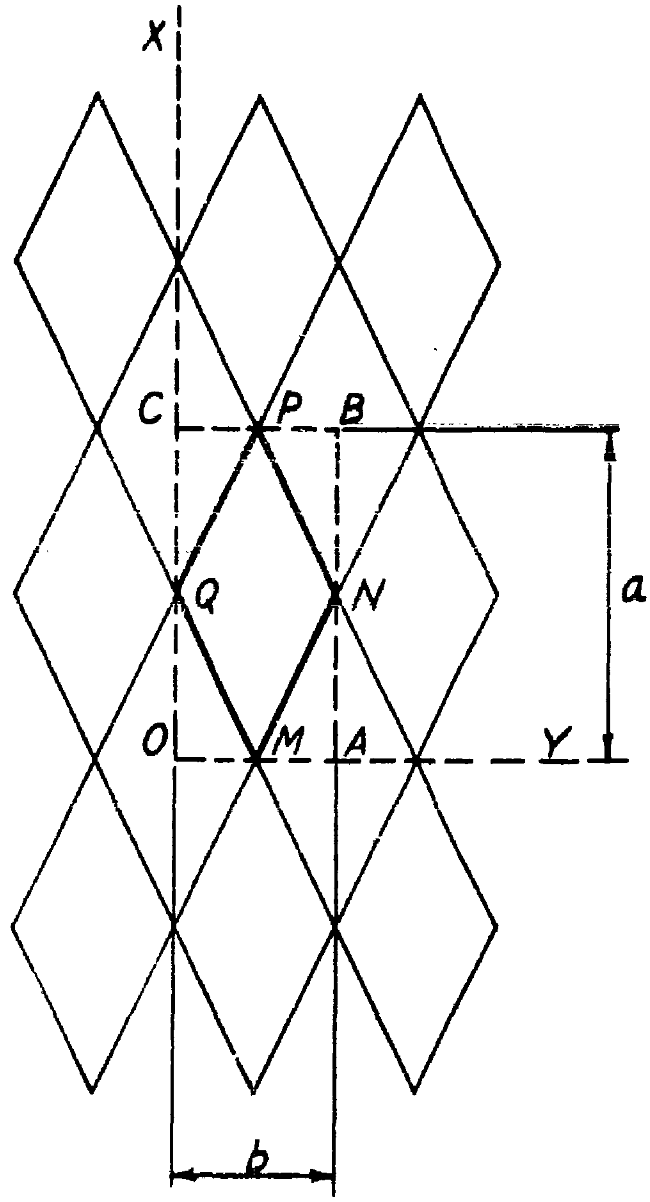

Figure 2. --Nodal lines of assumed diamond-shaped buckling pattern.

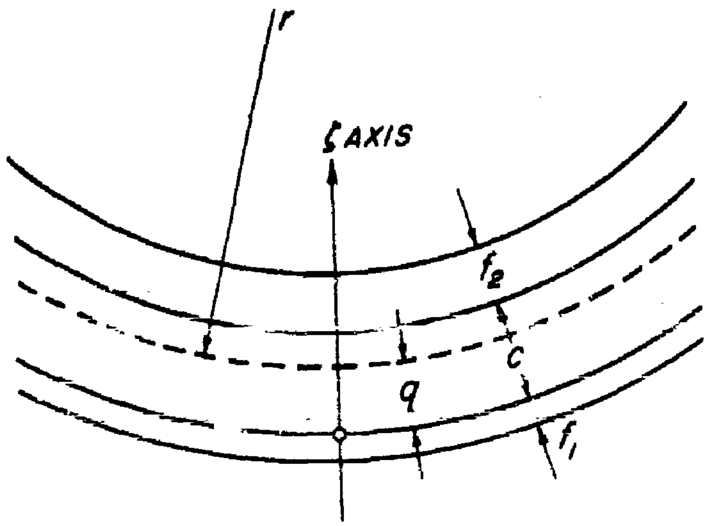

Figure 3. --Section of a cylindrical shell, where r is the radius of the middle surface of the shell, c is the thickness of the core, f_1 and f_2 are the thicknesses of the facings, q is the distance indicated in the figure, and ζ is the coordinate indicated in the figure.
Note: The curved lines are arcs of concentric circles.

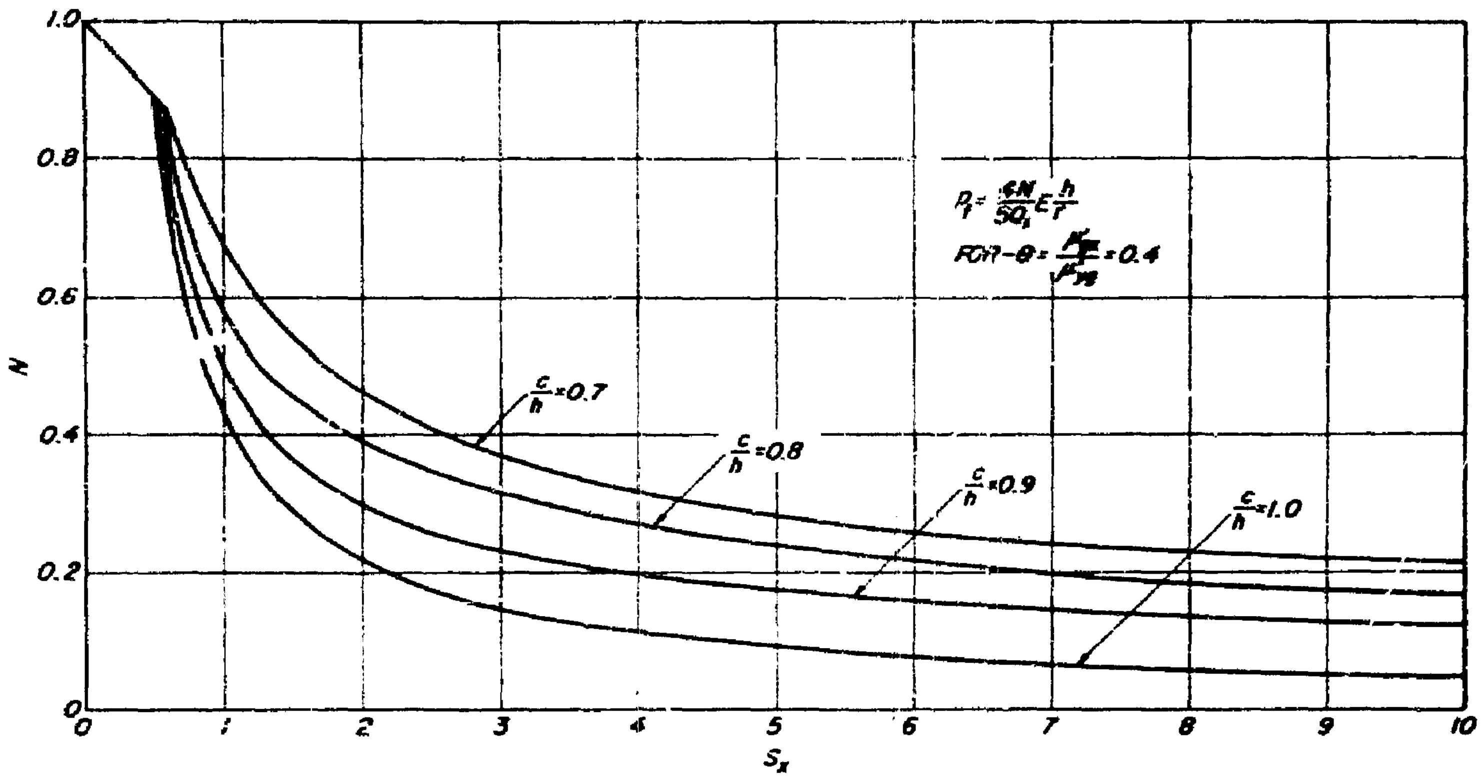

Figure 4. --$\underline{N}$ values for buckling of cylinders of sandwich constructions in axial compression. Isotropic sandwich facings of equal thickness, orthotropic core.

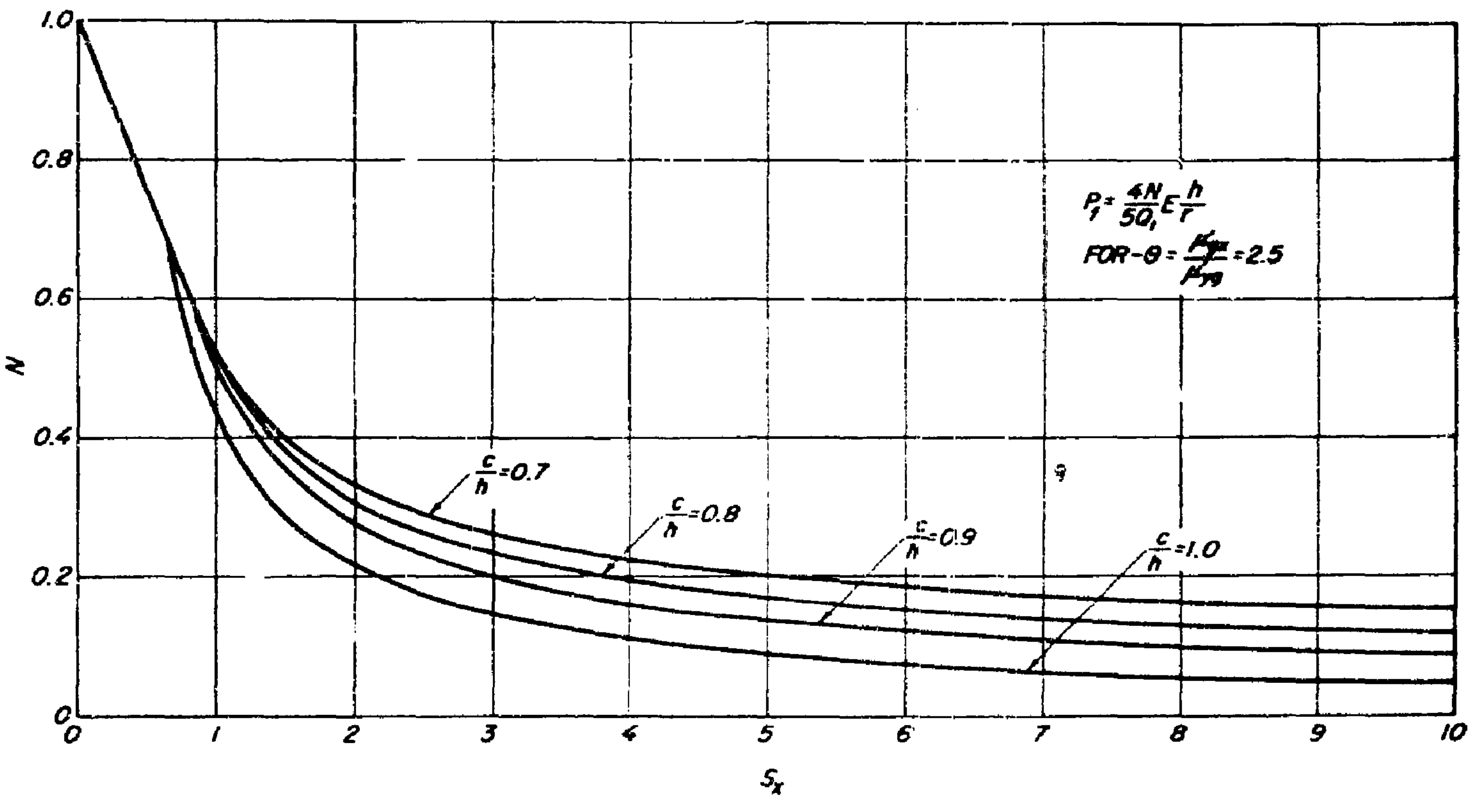

Figure 5. --$\underline{N}$ values for buckling of cylinders of sandwich constructions in axial compression. Isotropic sandwich facings of equal thickness, orthotropic core.

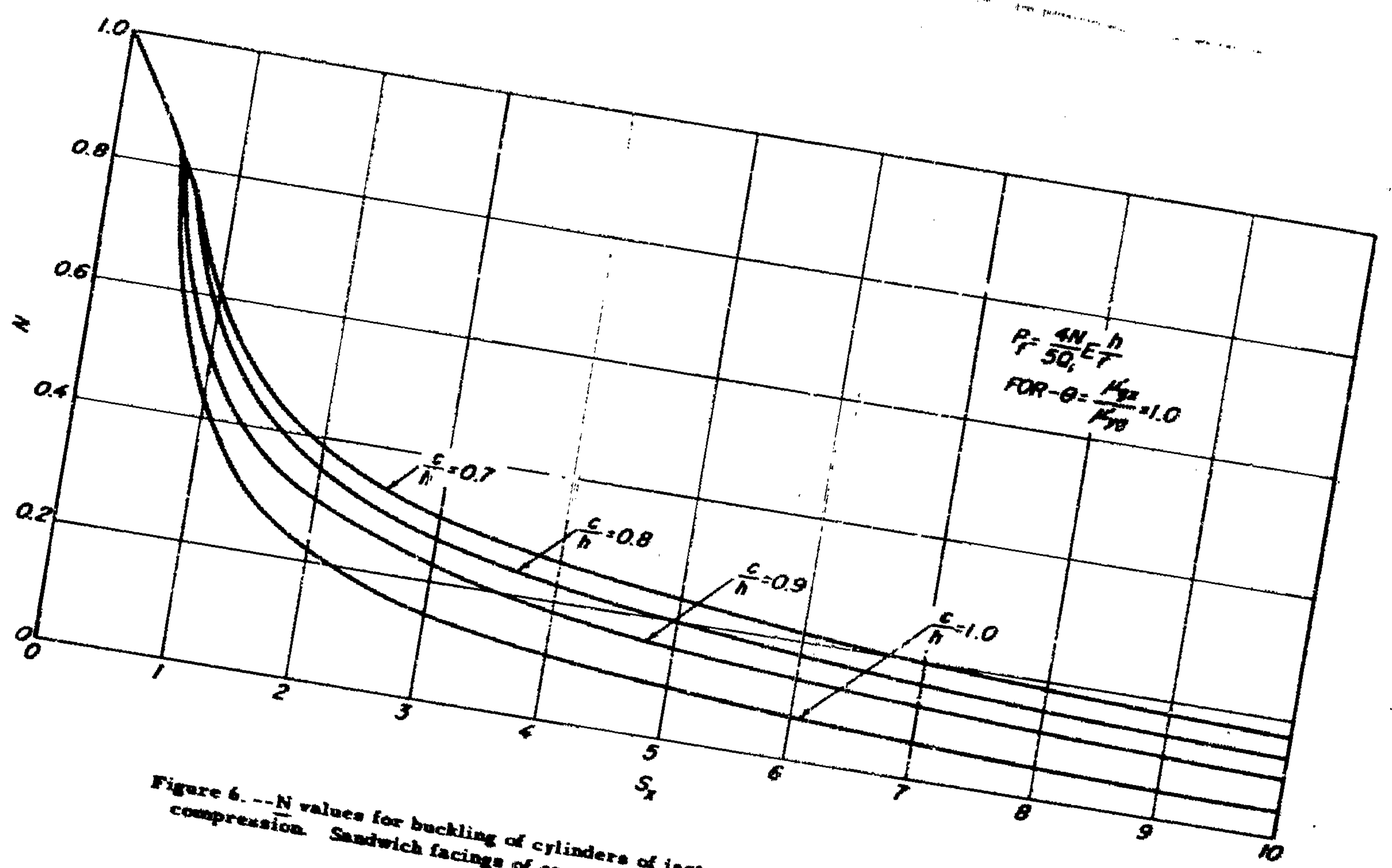

Figure 6. --N values for buckling of cylinders of isotropic sandwich constructions in axial compression. Sandwich facings of equal thickness.

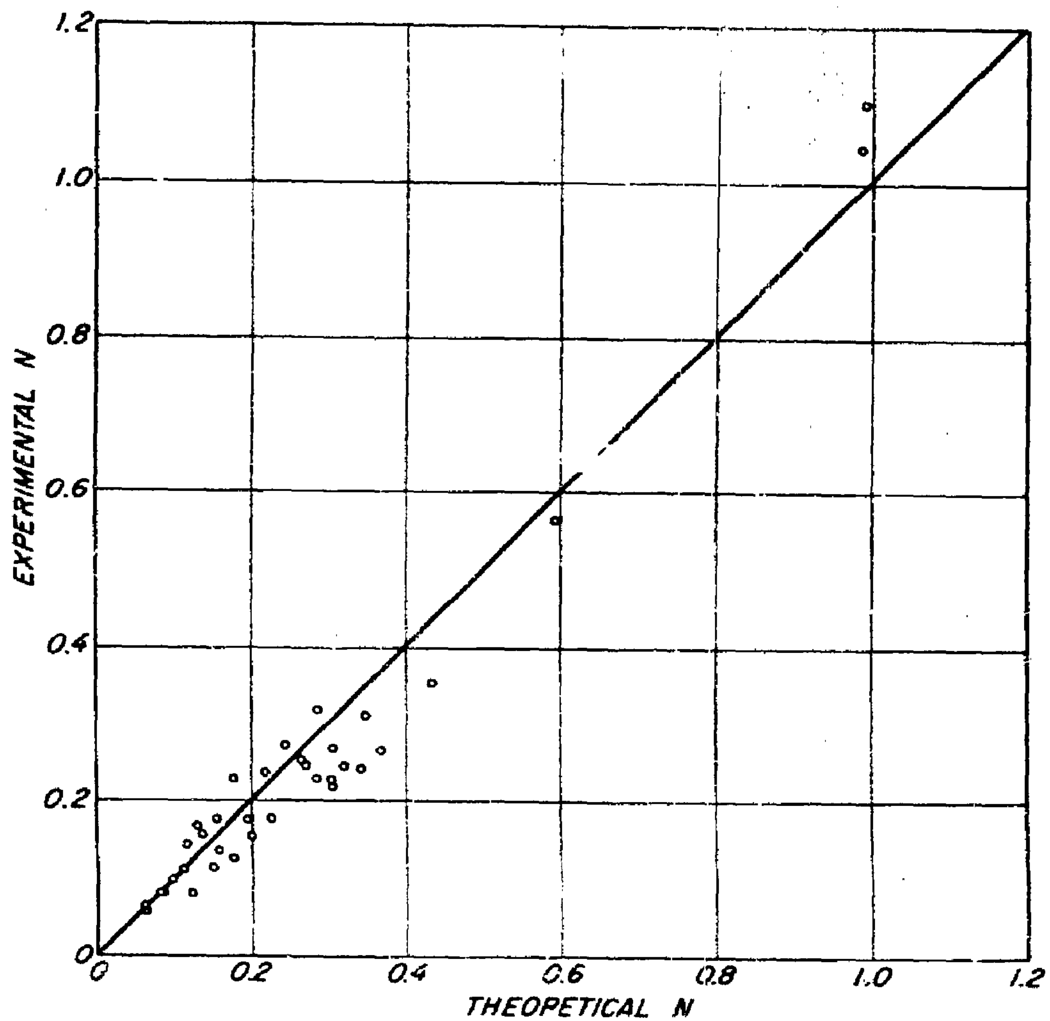

Figure 7. --Comparison of experimental values of $\underline{N}$ with theoretical values.

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